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CONNECTEDNESS OF CERTAIN RANDOM GRAPHS

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Dedicated to Aryeh Dvoretzky

ABSTRACT

L. Dubins conjectured in 1984 that the graph on vertices $\{1, 2, 3, \ldots\}$ where an edge is drawn between vertices i and j with probability $p_{ij} = \lambda/\max(i, j)$ independently for each pair i and j is a.s. connected for $\lambda = 1$. S. Kalikow and B. Weiss proved that the graph is a.s. connected for any $\lambda > 1$. We prove Dubin's conjecture and show that the graph is a.s. connected for any $\lambda > \frac{1}{4}$. We give a proof based on a recent combinatorial result that for $\lambda \leqq \frac{1}{4}$ the graph is a.s. disconnected. This was already proved for $\lambda < \frac{1}{4}$ by Kalikow and Weiss. Thus $\lambda = \frac{1}{4}$ is the critical value for connectedness, which is surprising since it was believed that the critical value is at $\lambda = 1$.

§1. Introduction

In an elegant paper [KW], S. Kalikow and B. Weiss made a significant extension of the now-classical theory [ER], [B] of connectedness of finite random graphs to a class of infinite random graphs. The interesting class of infinite random graphs are those on a countably infinite vertex set N where each edge is drawn randomly and independently with probability p_{ij} given for each pair of vertices i and j in N, and where $0 \leqq p_{ij} < 1$ satisfy the basic condition that

$$(1.1) \qquad \sum_{i \in A} \sum_{j \notin A} p_{ij} = \infty \qquad \text{for every proper subset } A \text{ of } N,$$

which, by the Borel–Cantelli lemma, says that A and A^c are connected with

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probability 1 for every subset A of N. Of course there are uncountably many A's so (1.1) does not imply connectedness.

Under (1.1), the fundamental dichotomy of Kalikow and Weiss [KW] says that the event that the graph is connected has probability either 0 or 1. Moreover, when it is not connected, they show it has a.s. infinitely many components. The general problem is to decide which of the two possibilities holds for a given p_{ij} satisfying (1.1). It seems difficult to give a necessary and sufficient condition on p_{ij} for connectedness under (1.1). We remark that under (1.1) a necessary condition for connectedness is that

$$(1.2) \qquad Ev_{ij} = \infty \qquad \text{for every } i \in N, \quad j \in N, \quad i \neq j$$

where v_{ij} = the number of self-avoiding paths from i to j in the graph. It seems possible that (1.2) is also sufficient under (1.1).†

Many results are known [B] about connectedness of graphs in the finite case with equal edge probabilities and these results form the basis of the techniques used here and in [KW] and are due to Erdos and Renyi [ER]. Since it appears difficult to give necessary and sufficient conditions on p_{ij} for connectedness to hold in the general case of (1.1), it is reasonable to ask about specific choices of p_{ij}'s. The class of interest here and in [KW] depends on a parameter λ and is given for $0 < \lambda < 2$ by

$$(1.3) \qquad p_{ij} = p_{ij}(\lambda) = \lambda / \max(i, j), \qquad i, j \in N = \{1, 2, \ldots\}.$$

Such random graphs satisfy (1.1) for all λ and are interesting because it was shown in [KW] that for $\lambda > 1$ connectedness holds, while for $\lambda < \frac{1}{4}$ disconnectedness holds. Since the probability of connectedness is clearly monotonically increasing in λ, there is by the fundamental dichotomy theorem [KW] a critical λ_0 so that for $\lambda > \lambda_0$ the graph is connected while for $\lambda < \lambda_0$ the graph is disconnected a.s. Thus [KW] proved that $\frac{1}{4} \leqq \lambda_0 \leqq 1$ and it was conjectured that $\lambda_0 = 1$ is the actual value.

Lester Dubins had conjectured long ago that $\lambda = 1$ was a case of connectedness. We show in §2 that the critical value is $\lambda_0 = \frac{1}{4}$ so that Dubin's conjecture is true (with room to spare). We show in §3 that (1.2) holds if and only if $\lambda > \frac{1}{4}$, which indicates that it may be true that (1.2) is equivalent to connectedness.† Although (1.2) is not an easy condition to use, it is easier than directly proving connectedness as seen in §3.

† *Added in proof.* R. Durrett has given a counterexample to this, to appear.

REMARK. Perhaps because the critical value was thought to be at $\lambda = 1$, [KW] pointed out the analogy to the fact that the critical value is $\lambda = 1$ in an apparently unrelated problem, namely that of deciding for which $0 < \lambda < 2$ arcs of length λ/n, $n = 2, 3, \ldots$ cover a unit circumference C under random rotations. [KW] refer to [S] (see [K, ch. 11] for a more readable proof) where it is shown that the arcs cover C infinitely often with probability 0 or 1 according as $\lambda < 1$ or $\lambda \geqq 1$. Despite that $\lambda_0 = \frac{1}{4}$ in the connectedness problem and $\lambda_0 = 1$ in the covering problem, the two problems are rather more directly related as follows. Namely if $A \subset N$ is a component of the graph then there is no link between A and A^C (since there are uncountably many $A \subset N$ this can occur for some A even though it has probability 0 for each fixed A by (1.1)). Similarly each fixed point $x \in C$ is covered with probability one by the arcs but since C is uncountable some point may not be covered. The analogy is actually much stronger: The connectedness problem is *exactly* equivalent to a covering problem by a random union U of subsets B_{ij} of $I = (0, 1)$. Let

$$B_{ij} = \{x \in (0, 1) : x_i \neq x_j\} \tag{1.4}$$

where $x = .\,x_1x_2\ldots$ is the binary expansion of x (the set of x where this is ambiguous is countable and does not matter). Now include B_{ij} in the union U with probability p_{ij}. Then $U = I$ w.p.1 if and only if the graph with edge probabilities p_{ij} is connected. Indeed a subset A of N has no link to $N - A$, i.e. the graph is not connected if and only if $x = x_A = .\,x_1x_2\ldots$ where $x_i = \chi(i \in A)$ is not covered by U. The equivalence of the two problems is not useful because the methods of [S] and [K] break down when the covering sets are not intervals. The sets B_{ij} are far from intervals and have many holes.

We give in §3 a somewhat different proof of the theorem of Kalikow and Weiss [KW] that $\lambda_0 \geqq \frac{1}{4}$ based on an interesting combinatorial identity [DMOS]. Whereas [KW] prove that for $\lambda < \frac{1}{4}$ the graph is disconnected, this proof shows that it also is disconnected for $\lambda = \frac{1}{4}$. It is perhaps surprising that one can answer the question for every λ, even at the critical value. It would be interesting to consider other p_{ij}'s, e.g. $p_{ij} = \lambda(i + j)$ or $p_{ij} = \lambda/\sqrt{i^2 + j^2}$.†

§2. $\lambda_0 \leqq \frac{1}{4}$

We prove that if $\lambda > \frac{1}{4}$ then the graph is a.s. connected by sharpening the method of [KW]. Their method relies on the technique of Erdos and Renyi

† See footnote at the end of the paper.

[ER, B] to prove that if $p_{ij} \equiv c/n$ for a graph on $\{1, 2, \dots, n\}$ then if $c > 1$ there is a giant component, i.e. one whose size is a positive fraction of n. We sharpen the method by extending the technique of [ER] to finite graphs with non-constant edge probabilities using the method of Chebysheff, Esscher, Chernoff, Bahadur–Rao, Donsker–Varadhan, now called large deviation theory.

The following lemma is implicit in [KW].

LEMMA 2. *Suppose λ has the property that there exist $\varepsilon > 0$, $\gamma > 0$, $\delta > 0$ such that for large n the subgraph $G(n)$ on n vertices,*

$$\{\lfloor \varepsilon n \rfloor + 1, \lfloor \varepsilon n \rfloor + 2, \dots, \lfloor \varepsilon n \rfloor + n\}$$

with edge probabilities $p_{ij} = \lambda/\max(i, j)$ has maximum component of size at least γn with probability at least δ. Then the graph is connected, i.e., $\lambda_0 \geqq \lambda$.

PROOF (after [KW]). Consider any $i < \lfloor \varepsilon n \rfloor$. The chance that i is linked directly to some element of the maximum component of the subgraph $G(n)$ is at least

$$\left[1 - \left(1 - \frac{1}{n(1+\varepsilon)}\right)^{\gamma n}\right] \approx e^{-\gamma/(1+\varepsilon)} \triangleq \theta$$

given that the maximum component of $G(n)$ is at least of size γn. Thus any pair i and j each less than $\lfloor \varepsilon n \rfloor$ are connected to each other via the maximum component of $G(n)$ with probability at least $\delta \cdot \theta^2$. But we can choose an infinity of disjoint subgraphs $G(n_k)$ by choosing $n_{k+1} > n_k(\varepsilon + 1)/\varepsilon$, $k = 1, 2, \dots$, and i and j are independently linked to each other through the maximum component of $G(n_k)$ with probability at least $\delta \cdot \theta^2$ for each k. Thus i and j are linked with probability one. Since there are only countably many pairs (i, j) we are done. ■

It remains only to prove that if $\lambda \geqq \frac{1}{4}$ then the hypothesis of the lemma holds, i.e. for large n, the maximum component of $G(n) = \{\lfloor \varepsilon n \rfloor + 1, \dots, \lfloor \varepsilon n \rfloor + n\}$ is of size at least γn with probability at least δ.

Choose an integer $L \geqq 1$ and consider the graph G on $\{1, 2, \dots, n\}$ where n is a multiple of L and where the edge probabilities are for $i, j \in \{1, \dots, n\}$ where n is a multiple of L and where the edge probabilities are for $i, j \in \{1, \dots, n\}$,

$$(2.1) \qquad p'_{ij} = \pi_l \triangleq \frac{\lambda}{\left(\varepsilon + \frac{l}{L}\right)} \frac{1}{n} \quad \text{if } \max(i, j) \in B_l \triangleq \left\{\frac{l-1}{L} n + 1, \dots, \frac{l}{L} n\right\}.$$

It is clear by monotonicity, or coupling, that the maximum component of $G(n)$ is stochastically at least as large as that of G since it is easy to see that

$$p'_{ij} \leqq p_{\lfloor \varepsilon n \rfloor + i, \lfloor \varepsilon n \rfloor + j} \qquad \text{for all } i \text{ and } j \in \{1, \ldots, n\}, \tag{2.2}$$

and increasing the number of edges can only increase the size of the maximum component. We need to show that G has a giant component.

We first seed, or start off, a large component. Thus suppose a_{l0}, $l = 1, \ldots, L$ are arbitrary but fixed integers. We first show that we can choose $\delta > 0$ and $0 < \gamma < 1/(2L)$ so that for large n, the subgraph G' of G consisting of the union of B'_l, the first $M = \lfloor n\gamma \rfloor$ elements of each block B_l,

$$B'_l \triangleq \left\{\frac{l-1}{L} n + 1, \ldots, \frac{l-1}{L} n + M\right\} \subset B_l, \; l = 1, \ldots, L, \; M = \lfloor n\gamma \rfloor \tag{2.3}$$

has at least a_{l0} elements each joined to element 1 of G by an edge, with probability at least δ.

To see this note that the number of elements of B'_l linked to 1 by an edge with edge probabilities p'_{ij} in (2.1) is asymptotically Poisson as $n \to \infty$ with parameter $\pi_l \cdot \gamma_n = \lambda\gamma/(\varepsilon + l/L)$. Since this is a fixed number and a_{l0}, $l = 1, \ldots, L$ are fixed this will have some positive probability, call it δ, for all large n for any fixed λ, γ, ε, L.

Now let A_{l0} be the actual set of elements of B'_l and note that the union of A_{l0}, $l = 1, \ldots, L$ are all connected to 1 and hence connected. If $A_{l0}, A_{l1}, \ldots, A_{lk}$, $l = 1, \ldots, L$ have been defined for a $k \geqq 0$, define $A_{l,k+1}$ as the set of elements of $B_l - B'_l$ which are directly linked by an edge to some element of $A_{1k} \cup A_{2k} \cup \cdots \cup A_{Lk}$ and which are not already in any of $A_{l0}, A_{l1}, \ldots, A_{lk}$. In other words, $A_{l,k+1}$ are those elements of $B_l - B'_l$ which are connected by a path of length $k + 1$ but not by a path of smaller length to some element of $A_{10} \cup A_{20} \cup \cdots \cup A_{L0}$. Denote $|A_{lk}|$ by a_{lk}, $l = 1, \ldots, L$, $k \geqq 0$.

If for some l and some k

$$a_{l0} + a_{l1} + \cdots + a_{lk} \geqq \gamma n \tag{2.4}$$

then there is a giant component because the maximum component exceeds that of the component of the element 1 which is already a positive fraction γ of n if (2.4) holds for some l and k. However, if (2.4) fails for k and each l, then we show that the process can be continued to stage $k + 1$ with high probability and so on until (2.4) does hold with positive probability.

To see this, suppose there exists a $\theta > 1$ and a vector $(\xi_1, \dots, \xi_L)$ with positive entries such that for $k' < k$,

$$\sum_{l=1}^{L} \xi_l a_{l,k'+1} > \theta \sum_{l=1}^{L} \xi_l a_{l,k'}. \tag{2.5}$$

We will actually choose $\xi_l = 1/\sqrt{l}$, $l = 1, \dots, l$ for a sufficiently large L. We want to show that with high probability (2.5) continues to hold for $k' = k$. Now given $a_{l0}, \dots, a_{lk}$ for $l = 1, 2, \dots, L$ and M, $a_{1,k+1}$ is a random variable conditionally stochastically equal to the sum of $n/L - M - a_{11} - a_{12} - \cdots - a_{1k}$ independent Bernoulli variables with success probability from (2.1) given by

$$P_1 \triangleq 1 - (1-\pi_1)^{a_{1k}}(1-\pi_2)^{a_{2k}} \cdots (1-\pi_L)^{a_{Lk}} \approx \sum_{l'=1}^{L} \frac{\lambda}{\varepsilon + l'/L} a_{l'k} \frac{1}{n}. \tag{2.6}$$

For general $l \geqq 1$, $a_{l,k+1}$ is a random variable which is conditionally stochastically equal to the sum of $n/L - M - a_{l1} - a_{l2} - \cdots - a_{lk}$ independent Bernoulli variables with success probability from (2.1) given by

$$\begin{aligned} p_l &\triangleq 1 - (1-\pi_l)^{a_{1k} + \cdots + a_{lk}}(1-\pi_{l+1})^{a_{l+1,k}} \cdots (1-\pi_L)^{a_{Lk}} \\ &\approx \left(\frac{\lambda}{\varepsilon + l/L} (a_{lk} + \cdots + a_{lk}) + \sum_{l'=l+1}^{L} \frac{\lambda}{\varepsilon + l'/L} a_{l'k} \right) \frac{1}{n}. \end{aligned} \tag{2.7}$$

Since $M < \gamma n$ and (2.4) holds for k,

$$\frac{n}{L} - M - a_{l1} - \cdots - a_{lk} \geqq n\left(\frac{1}{L} - 2\gamma\right) \tag{2.8}$$

so that $a_{l,k+1}$ is conditionally stochastically larger than the sum of $n(1/L - 2\gamma)$ independent Bernoulli variables with success probability as in (2.7), $l = 1, \dots, L$. Since $a_{l,k+1}$, $l = 1, \dots, L$ are independent, by (2.7), (2.8), and Chernoff's inequality, for any $\alpha > 0$ we have with p_l as in (2.7),

$$\begin{aligned} &P\left(\theta \sum_{l=1}^{L} \xi_l a_{lk} > \sum_{l=1}^{L} \xi_l a_{l,k+1} \,\middle|\, a_{l0}, \dots, a_{lk}, l = 1, \dots, L, M\right) \\ &\leqq E\left(e^{\alpha(\theta \sum_{l=1}^{L} \xi_l a_{lk} - \sum_{l=1}^{L} \xi_l a_{l,k+1})} \,\middle|\, a_{l0}, \dots, a_{lk}, l = 1, \dots, L, M\right) \\ &\leqq e^{\alpha\theta \sum_{l=1}^{L} \xi_l a_{lk}} \prod_{l=1}^{L} (e^{-\alpha\xi_l} p_l + 1 - p_l)^{n(1/L - 2\gamma)}. \end{aligned} \tag{2.9}$$

Since $1-x \leqq \exp(-x)$ we have that the rhs of (2.9) is less than

$$\exp\left(\alpha\theta \sum_{l=1}^{L} \xi_l a_{lk} - n\left(\frac{1}{L}-2\gamma\right)\sum_{l=1}^{L}(1-e^{-\alpha\xi_l})p_l\right) \leqq e^{-\eta(a_{1k}+\cdots+a_{Lk})} \tag{2.10}$$

for some $\eta > 0$, provided that for small positive α the exponent on the left in (2.10) is less than that on the right of (2.10). For this we need that

$$\theta \sum_{l=1}^{L} \xi_l a_{lk} - n\left(\frac{1}{L}-2\gamma\right)\sum_{l=1}^{L} \xi_l p_l < -\eta(a_{lk}+\cdots+a_{Lk}). \tag{2.11}$$

Putting in the approximation to p_l given in (2.7) we need

$$\begin{aligned} &\theta \sum_{l=1}^{L} \xi_l a_{lk} - \lambda \sum_{l=1}^{L} \xi_l \left\{\frac{a_{lk}+\cdots+a_{lk}}{\varepsilon + l/L} + \sum_{l'=l+1}^{L} \frac{a_{l'k}}{\varepsilon + l'/L}\right\}\left(\frac{1}{L}-2\gamma\right) \\ &< -\eta(a_{lk}+\cdots+a_{Lk}). \end{aligned} \tag{2.12}$$

In order that we can find $\theta > 1$, $\varepsilon > 0$, $\eta > 0$ such that (2.12) holds for all $a_{lk}, \ldots, a_{Lk}$ it is necessary and sufficient that the coefficient of a_{lk} on the left of (2.12) is less than the coefficient of a_{LK} on the right for $l = 1, \ldots, L$. That is we must have for $l = 1, \ldots, L$, for some $\theta > 1$, $\eta > 0$, $\varepsilon > 0$,

$$\theta\xi_l - \frac{\lambda}{L}\left[\sum_{l'=l}^{L} \frac{\xi_{l'}}{\varepsilon + l'/L} + \frac{1}{\varepsilon + l/L}\sum_{l'=1}^{l-1} \xi_{l'}\right] < -\eta. \tag{2.13}$$

Since $\theta > 1$, $\varepsilon > 0$, $\eta > 0$ are otherwise arbitrary we must require

$$\xi_l < \lambda\left[\sum_{l'=l}^{L} \frac{\xi_{l'}}{l'} + \frac{1}{l}\sum_{l'=1}^{l-1} \xi_{l'}\right], \qquad l = 1, \ldots, L. \tag{2.14}$$

This must hold for some positive $\xi_1, \ldots, \xi_L$, and if it does, then it will follow from (2.9) that

$$\begin{aligned} &P\left(\sum_{l=1}^{L} \xi_l a_{l,k+1} < \theta \sum_{l=1}^{L} \xi_L a_{l,k} \,\middle|\, a_{l0}, \ldots, a_{lk}, l = 1, \ldots, L, M\right) \\ &\leqq e^{-\eta(a_{1k}+\cdots+a_{Lk})}. \end{aligned} \tag{2.15}$$

But then with the remaining probability we will have

$$\sum_{l=1}^{L} \xi_l a_{l,k+1} > \theta \sum_{l=1}^{L} \xi_l a_{l,k} \tag{2.16}$$

and so (2.5) continues and so, as long as k is such that (2.4) holds,

$$\sum_{l=1}^{L} \xi_l a_{l,k} > \theta^k \sum_{l=1}^{L} \xi_l a_{l,0} > \theta^k \min_{1 \leqq l \leqq L} \xi_l \sum_{l=1}^{L} a_{l,0}. \tag{2.17}$$

Since $\theta > 1$ this says that $\Sigma \xi_l a_{l,k}$ is large and since $\xi_l > 0$ for $l = 1, \ldots, L$, we must have that $a_{lk} > 0$ for some $l = 1, \ldots, L$ and the process continues until (2.4) holds on a set of positive probability. This probability is positive because the upper bound (2.15) forms the kth term of the series

$$\sum_{k=0}^{\infty} e^{-\eta\theta^k(\min_{1 \leq l \leq L} \xi_l / \max_{1 \leq l \leq L} \xi_l) \Sigma_{l=1}^{L} a_{l,0}} \tag{2.18}$$

which sums to less than 1 because of the last bound in (2.17) and the fact that $a_{l,0}$, $l = 1, \ldots, L$ can be chosen large. Thus we need to show that if $\lambda > \frac{1}{4}$ then positive ξ_l in (2.14) can be found. But (2.14) is equivalent to the maximum eigenvalue of the matrix $A_{ll'} = 1/\max(l, l')$, $1 \leqq l, l' \leqq L$, being larger than $1/\lambda$. We will show that as $L \to \infty$ this maximum eigenvalue is at least 4, so that if $\lambda > \frac{1}{4}$ there will exist an L such that $\xi_1, \ldots, \xi_L$ exists to make the proof work.

Indeed A is a symmetric matrix with positive entries whose largest eigenvalue is positive by Frobenius's theorem. By the Weyl–Courant lemma, [RN, p. 237] the largest eigenvalue is given by

$$\max_{\xi \neq 0} \frac{(A\xi, \xi)}{(\xi, \xi)} \geqq \frac{(A\xi', \xi')}{(\xi', \xi')} \tag{2.19}$$

where

$$\xi'_l = \frac{1}{\sqrt{l}}, \qquad l = 1, \ldots, L. \tag{2.20}$$

Since

$$\begin{aligned} (A\xi', \xi') &= \sum_{l=1}^{L} \sum_{l'=1}^{L} \frac{1}{\sqrt{l}\sqrt{l'} \max(l, l')} \\ &= \sum_{l=1}^{L} \frac{1}{l^{3/2}} \sum_{l'=1}^{l-1} \frac{1}{l'} + \sum_{l=1}^{L} \frac{1}{\sqrt{l}} \sum_{l'=l}^{L} \frac{1}{l'^{3/2}} \qquad (2.21) \\ &= 4 \sum_{l=1}^{L} \frac{1}{l} + O(1) \end{aligned}$$

while

$$(\xi', \xi') = \sum_{l=1}^{L} \frac{1}{(\sqrt{l})^2} = \sum_{l=1}^{L} \frac{1}{l} \tag{2.22}$$

we see from (2.19) that the maximum eigenvalue of A is asymptotically at least 4. It can be shown directly that the limiting maximum eigenvalue is 4 but we do not need this and omit it.

The proof is complete.

§3. $\lambda_0 \geqq \frac{1}{4}$

We give a new proof that if $\lambda < \frac{1}{4}$ then the graph is a.s. disconnected which is similar to the proof of [KW] but is slightly tighter and enables one to show that $\lambda = \frac{1}{4}$ is also a case of disconnectedness. We need the following lemma.

LEMMA 2. *If v_{ij} is the number of self-avoiding paths from i to j in a graph satisfying* (1.1) *and if for some $i \neq j$,*

$$Ev_{ij} < \infty, \tag{3.1}$$

then the graph is a.s. disconnected.

PROOF. If $Ev_{ij} < \infty$, then by replacing a finite number of p_{ij}'s by zero (call the new p_{ij}'s, p'_{ij}) we can make $Ev'_{ij} < 1$ where s' refers to p'_{ij}. But then

$$P(v'_{ij} > 0) \leqq Ev'_{ij} < 1 \tag{3.2}$$

and so the graph with p'_{ij} has probability less than one of being connected. But the same must then be true for the original p_{ij} since a finite number of Bernoulli edge choices has positive probability to produce all failures or non-edges. By the fundamental theorem [KW] the probability that the original graph is connected must be zero since it is < 1. ■

We next give a formula for Ev_{ij}. This is slightly neater if we add the vertex 0 to $N = \{1, 2, \ldots\}$ keeping the same rule (1.3) for p_{ij}. Then the expected number of self-avoiding paths from vertex 0 to vertex 1 is

$$\begin{aligned} Ev_{01} = p_{01} + \sum_{k \geqq 1} \sum_{\sigma \in \mathscr{S}k} \sum_{1 < s_1 < \cdots < s_k} & p(0, s_{\sigma_1}) p(s_{\sigma_1}, s_{\sigma_2}) \\ & \cdots p(s_{\sigma_{k-1}}, s_{\sigma_k}) p(s_{\sigma_k}, 1) \end{aligned} \tag{3.3}$$

where the sum is over all $k \geqq 1$ and all $k + 1$-step paths from 0 to 1 which visit distinct vertices $s_1 < s_2 < \cdots < s_k$ before visiting 1 in some permuted order $\sigma \in \mathscr{S}_k$, the set of permutations on $\{1, \ldots, k\}$. Using (1.3) we get for $p_{ij} = \lambda / \max(i, j)$,

$$(3.4)\quad \begin{aligned} Ev_{01} &= \frac{\lambda}{1} + \sum_{k\geq 1} \sum_{\sigma\in\mathcal{S}_k} \sum_{1<s_1<\cdots<s_k} \frac{\lambda}{s_{\sigma_1}} \frac{\lambda}{\max(s_{\sigma_1}, s_{\sigma_2})} \cdots \frac{\lambda}{\max(s_{\sigma_{k-1}}, s_{\sigma_k})} \frac{\lambda}{s_{\sigma_k}} \\ &= \lambda + \sum_{k\geq 1} \lambda^{k+1} \sum_{\sigma\in\mathcal{S}_k} \sum_{1<s_1<\cdots<s_k} \frac{1}{s_1^{\varepsilon_1(\sigma)} s_2^{\varepsilon_2(\sigma)} \cdots s_k^{\varepsilon_k(\sigma)}} . \end{aligned}$$

The powers $\varepsilon_j(\sigma)$, $j = 1, \ldots, k$ in (3.4) are either 0, 1, 2, where if we let $\sigma_0 = s_0 = 0$, and $\sigma_{k+1} = -1$, and $s_{-1} = 1$ then for $1 \leqq j \leqq k$,

$$(3.5)\quad \varepsilon_j(\sigma) = \begin{cases} 0 & \text{if } \sigma_l = j \text{ and neither of } \sigma_{l-1} \text{ and } \sigma_{l+1} \text{ is } < j, \\ 1 & \text{if } \sigma_l = j \text{ and exactly one of } \sigma_{l-1} \text{ and } \sigma_{l+1} \textit{ is } < j, \\ 2 & \text{if } \sigma_l = j \text{ and both of } \sigma_{l-1} \text{ and } \sigma_{l+1} \text{ is } < j. \end{cases}$$

Comparing a sum with an integral it is easy to see that for $r > 0$ and $\varepsilon > 1$,

$$(3.6)\quad \sum_{s=r+1}^{\infty} \frac{1}{s^{\varepsilon}} \leqq \frac{1}{\varepsilon - 1} \frac{1}{r^{\varepsilon - 1}} .$$

Using (3.6) repeatedly in (3.4) we get a bound on Ev_{01},

$$(3.7)\quad \begin{aligned} Ev_{01} &\leqq \lambda + \sum_{k\geq 1} \lambda^{k+1} \sum_{\sigma\in\mathcal{S}_k} \sum_{1<s_1<\cdots<s_{k-1}} \frac{1}{s_1^{\varepsilon_1(\sigma)} s_2^{\varepsilon_2(\sigma)} \cdots s_{k-1}^{\varepsilon_{k-1}(\sigma)} (\varepsilon_k(\sigma) - 1) s_{k-1}^{\varepsilon_k(\sigma) - 1}} \\ &\leqq \lambda + \sum_{k\geq 1} \lambda^{k+1} \sum_{\sigma\in\mathcal{S}_k} \frac{1}{\xi_1(\sigma)\xi_2(\sigma)\cdots\xi_k(\sigma)} \end{aligned}$$

where

$$(3.8)\quad \xi_j(\sigma) = \varepsilon_k(\sigma) + \varepsilon_{k-1}(\sigma) + \cdots + \varepsilon_{k-j+1}(\sigma) - j, \quad j = 1, \ldots, k.$$

The variables $\xi_j(\sigma)$ may be considered as random variables on $\mathcal{S}_k$ with uniform distribution and then they have an interesting interpretation. Namely, since $\sigma_k^{-1}, \sigma_{k-1}^{-1}, \ldots, \sigma_{k-j+1}^{-1}$ are the indices which map under σ into the last j values in $\{1, \ldots, k\}$, $\xi_j(\sigma)$ is the number of islands present at time $j = 1, 2, \ldots, k$ among the ordered states $\{1, 2, \ldots, k\}$. Thus the interpretation of the variables $\xi_1, \xi_2, \ldots, \xi_k$ is that $\xi_1 = \xi_k = 1$ and if j balls have been dropped into exactly j of $k \geqq j$ adjacent urns in a row, then some of the urns containing balls will be contiguous and there will be a number, $\xi_j \geqq 1$, of islands of filled urns. For example, if $k = 9, j = 6$, and urns 2, 3, 4, 6, 8, 9 have been filled by the σ balls, then $\{2, 3, 4\}$, $\{6\}$, $\{8, 9\}$ are islands and $\xi_6 = 3$. In a companion paper [DMOS], the following remarkable theorem about $\xi_1, \ldots, \xi_k$ is proved:

THEOREM [DMOS].

$$E\frac{1}{\xi_1\cdots\xi_k}=\binom{2k}{k}\frac{1}{(k+1)!}\,. \tag{3.9}$$

Putting (3.9) into (3.7) we have

$$Ev_{01}\leqq\sum_{k\geqq 0}\lambda^{k+1}\binom{2k}{k}\frac{1}{k+1}\,. \tag{3.10}$$

Since

$$\binom{2k}{k}\approx\frac{2^{2k}}{\sqrt{\pi k}} \tag{3.11}$$

and $\Sigma k^{-3/2}<\infty$, we see that $Ev_{01}<\infty$ if and only if $\lambda\leqq\frac{1}{4}$. It follows from Lemma 2 that for $\lambda\leqq\frac{1}{4}$ the graph on $\{0,1,\ldots\}$ is a.s. disconnected and it follows by monotonicity, or coupling, that the original graph (1.3) is a.s. disconnected for $\lambda\leqq\frac{1}{4}$. Since

$$Ev_{ij}\leqq Ev_{01} \tag{3.12}$$

it is clear that (1.2) holds if and only if $\lambda>\frac{1}{4}$, so that it may be true that (1.2) is a necessary and sufficient condition for connectedness; at least it agrees on this class of examples.

REMARK. It is likely that the method of §2 could be used to at least partially treat other p_{ij} which are homogeneous of degree -1, e.g.[†]

$$p_{ij}=\lambda/(i+j). \tag{3.13}$$

We have not explored such extensions. Note that the form of $p_{ij}=\lambda/\max(i,j)$ was used very heavily in (3.4).

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[†] *Added in proof.* R. Durrett and H. Kesten have given an elegant extension of our results to this class of p_{ij}'s in *The critical parameter for connectedness of some random graphs*, manuscript. They give an exact formula for the critical value, $1/\pi$ in the case (3.13)!